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ASSIGNMENT PROBLEM

Definition:

An assignment problem is a particular case of transportation problem where the objective is to assign a number of resources to an equal number of activities so as to minimize total time or total cost or maximize total profit of allocation.

Given n facilities, n jobs and the effectiveness of each facility to each job, here the problem is to assign each facility to one and only one job so that the measure of effectiveness is optimized. Here the optimization means Maximization or Minimization. For example, the head of the department may have 6 people available for assignment and 6 jobs to fill so he may like to know which job should be assigned to which person so that all tasks can be done in the minimum possible time. The problems of this kind are known as assignment problem.

Mathematical Formulation of an Assignment Problem

The assignment problem can be stated in the form of (n x n) matrix known as cost or effectiveness matrix denoted by $[C_{ij}]$.

Where C_{ij} is the cost, if the ith person is assigned to the jth job.

$$\text{Persons}\begin{array}{c} \\ \end{array}\overset{\text{Jobs}}{\begin{array}{c|cccccc} & 1 & 2 & 3 & & j & n \\ \hline 1 & C_{11} & C_{12} & C_{13} & \cdots & C_{ij} & \cdots C_{1n} \\ 2 & C_{21} & C_{22} & C_{23} & \cdots & C_{2j} & \cdots C_{2n} \\ 3 & C_{31} & C_{32} & C_{33} & \cdots & C_{3j} & \cdots C_{3n} \\ i & C_{11} & C_{12} \cdots & C_{13} & \cdots & C_{Cj} & \cdots C_{in} \\ n & C_{n1} & C_{n2} \cdots & C_{n3} & \cdots & C_{nj} & \cdots C_{nn} \end{array}}$$

Mathematically an assignment problem can be stated as follows
Minimise the total cost

$$Z = \sum_{i=i}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

where $x_{ij} = 1$, if i^{th} person is assgned to the j^{th} job
0, if i^{th} person is that assigned to the j^{th} job

subject to the constraints

(i) $\sum_{i=1}^{n} x_{ij} = 1, j = 1,2 \text{ --- } n$

which means that only one job is done by the i-th person, i =1,2 ---n

(ii) $\sum_{j=1}^{n} x_{ij} = 1, i = 1,2\ n$

which means that only one person should be assigned to the j^{th} job, $j = 1,2$ --- n

Method of Finding an Optimum Solution to a given Assignment Problem:

The Hungarian Method:

The Hungarian method also known as reduced matrix method is an efficient method which helps in finding an optimal solution to a given assignment problem without making direct comparison among all possible solutions. It was developed and published in 1955 by Harold Kuhn, who gave the name "Hungarian method" because the algorithm was largely based on the earlier works of two Hungarian mathematicians: Dénes Kőnig and Jenő Egerváry.

The Hungarian method is based on a principle where a given cost matrix is converted into reduced cost matrix in such a way that each row and each column contain at least one single zero, like this it becomes possible to make optimal assignments.

Procedure of solving an Assignment Problem

The Hungarian Method is discussed in the form of a series of computational steps as follows, when the objective function is that of minimization type 3

Step 1: From the given problem, find out the cost table. Note that if the number of origins is not equal to the number of destinations then a dummy origin or destination must be added.

Step 2: In each row of the table, find out the smallest cost element, subtract this smallest cost element from each element in that row so that there will be at least one zero in each row of the new table. This new table is known as First Reduced Cost Table (matrix).

Step 3: In each column of the table, find out the smallest cost element, subtract this smallest cost element from each element in that column. As a result of this, each row and column has at least one zero element. This new table is known as Second Reduced Cost Table (matrix).

Step 4: Now determine an assignment as follows:

(a) Examine the rows successively one by one until a row containing exactly one zero is found. Then assignment indicated by '□' is marked to that zero. Now cross (×) all the zeros in the column in which the assignment is made. This procedure should be adopted for each row.

(b) After each row is examined, same procedure is applied to column. Examine all columns until a column containing exactly one zero is found. Then make an assignment in that position and cross other zeros in the row in which the assignment was made.

(c) If for a row and for a column there are two or more zero and one can't be chosen by inspection, choose the assigned zero cell arbitrarily.

(d) The above procedures may be repeated until every zero element cell is either assigned (boxed) or crossed out.

Step 5: An optimum assignment is found, if the number of assigned cells is equal to the number of rows (and columns). In case we had chosen a zero cell arbitrarily, there may be an alternate optimum solution. If no optimum solution is found i.e. some rows or columns without an assignment then go to Step 6.

Step 6: Draw minimum number of vertical and horizontal lines covering all the zeros in the reduced matrix in step 4, in the following manner:

i. Mark check (√) to those rows where no assignment has been made.
ii. Examine the checked (√) rows. If any assigned zero cell occurs in these rows, check (√) the respective columns that contains those zeros.
iii. Examine the checked (√) columns. If any assigned zero cell occurs in those columns, check (√) the respective rows that contains those assigned zeros.
iv. The process may be repeated until now more rows or column can be checked.
v. Draw straight lines through all unchecked rows and through all checked columns.

Step 7: Examine those elements that are not covered by a line. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them. Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table.

Step 8: go to step 4 and repeat the procedure until the number of assignments becomes equal to the number of rows (columns). Like this each row/column has an assignment. Thus the current solution is an optimal solution.

Example 1: A workshop contains four different machines available for work on four jobs. The table given below shows the associated cost matrix. Find the minimum cost possible through optimal assignment of machines to jobs.

Machines

		1	**2**	**3**	**4**
	A	7	22	12	17
Jobs	**B**	12	27	2	12
	C	32	17	9	21
	D	14	22	21	17

Solution:

Step 1: Find the First Reduced Cost Matrix by selecting the smallest element of each row and subtract this smallest element from all the elements of that row (table 1).

Table 1: First reduced cost matrix (row operation)

Machines

		1	**2**	**3**	**4**
	A	0	15	5	10
Jobs	**B**	10	25	0	10
	C	23	8	0	12
	D	0	8	7	3

Step 2: Find the Second Reduced Cost Matrix by selecting the minimum element from each column and subtract this element from all the elements of that column (table 2).

Table 2: Second reduced cost matrix (column operation)

Machines

		1	**2**	**3**	**4**
	A	0	7	5	7
Jobs	**B**	10	17	0	7
	C	23	0	0	9
	D	0	0	7	0

Step 3: Determine an Assignment. Now examine the rows one by one until a row containing exactly one single zero is found. Here, row 1 of the table 2 has only one zero in cell (1, 1) so here, we can make an assignment (box) to this cell and cross out all other zeros in the column 1. Crossing other zero make no possibility of further assignments in that column. Now examine row 2, we find that it has one zero in cell (2, 3), make an assignment to this cell and cross out the other zeros in column 3. Row 3 has one single zero in column 2, cell (2, 2), so we make an assignment and cross other zeros in column 2. Finally, row 4 has single zero left in column 4, cell (4, 4), so make an assignment.

As all the zeros are either assigned (boxed) or crossed and the number of assignments equal to the number of rows (columns), the assignment shown in the table 3 is optimal.

Table 3: Optimal assignment

Machines

		1	**2**	**3**	**4**
	A	[0]	7	5	7
Jobs	**B**	10	17	[0]	7
	C	23	[0]	~~0~~	9
	D	~~0~~	~~0~~	7	[0]

Optimal Assignment is as follows:

Jobs	Machines	Cost
A	1	7
B	3	2
C	2	17
D	4	17

Thus, the optimal cost of assignment is: 7+2+17+17= Rs 43

Example 2: Solve the following assignment problem that will result in minimum man-hours needed.

Jobs

		1	2	3	4	5
	A	3	10	3	8	2
Men	**B**	7	9	8	7	2
	C	5	7	6	4	2
	D	5	3	8	4	2
	E	6	4	10	6	2

Solution:

Step 1: Find the First Reduced Matrix by selecting the smallest element of each row and subtract this smallest element from all the elements of that row (table 4).

Table 4: First reduced matrix (row operation)

Jobs

		1	2	3	4	5
	A	1	8	1	6	0
Men	**B**	5	7	6	5	0
	C	3	5	4	2	0
	D	3	1	6	2	0
	E	4	2	8	4	0

Step 2: Find the Second Reduced Matrix selecting the minimum element from each column and subtract this element from all the elements of that column (table 5).

Table 5: Second reduced matrix (column operation)

Jobs

		1	2	3	4	5
	A	0	7	0	4	0
Men	**B**	4	6	5	3	0
	C	2	4	3	0	0
	D	2	0	5	0	0
	E	3	1	7	2	0

Step 3: Determine an assignment. Row 1 has three zero and one can't be chosen by inspection, choose the assigned zero cell (1, 1) arbitrarily and cross the remaining zeros. Row 2 has a single zero in column 5. Make an assignment i.e. boxed (□) it and cross the zeros in column 5. Now column 4 has a single zero in row 3, make an assignment and cross the other zero. Column 2 has a single zero in row 4 now make an assignment.

Now we can see (table 6) there are no remaining zeros and row E and column 3 has no assignment. Thus the optimal solution is not reached at this stage and we proceed to the next step.

Table 6:

Jobs

		1	**2**	**3**	**4**	**5**
	A	[0]	7	0	4	0
Men	**B**	4	6	5	3	[0]
	C	2	4	3	[0]	0
	D	2	[0]	5	0	0
	E	3	1	7	2	0

Step 4: Draw minimum number of vertical and horizontal lines equal covering all the zeros in the reduced matrix (table 7), in the following manner

i. Mark check (√) to row 5, where no assignment has been made.

ii. Then mark check (√) to column 5, which has a zero in the marked row.

iii. Mark check (√) to row 2 which has assignment in the marked column.

iv. Draw lines through all unmarked rows and through all marked columns.

Table 7

Jobs

		1	**2**	**3**	**4**	**5**
	A	[0]	7	0	4	0
Men	**B**	4	6	5	3	[0]
	C	2	4	3	[0]	0
	D	2	[0]	5	0	0
	E	3	1	7	2	0

Step 5: Examine the elements that do not have a line through them. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them. Add this smallest element to every element that lies at the intersection of two lines. Then the resulting matrix is a new revised cost table (table 8).

Table 8

Jobs

		1	**2**	**3**	**4**	**5**
	A	[0]	7	0	4	1
Men	**B**	3	5	4	2	[0]
	C	2	4	3	[0]	1
	D	2	[0]	5	0	1
	E	2	0	6	1	0

Step 6: Repeat the steps to obtain optimal solution

Table 9

Jobs

		1	**2**	**3**	**4**	**5**
	A	~~0~~	9	[0]	6	3
Men	**B**	1	5	2	2	[0]
	C	[0]	4	1	~~0~~	1
	D	~~0~~	~~0~~	3	[0]	1
	E	~~0~~	[0]	4	1	~~0~~

In table 9 there are no remaining zeros and every row and column has assignment and optimal solution is reached.

Men	Jobs	Man hours
A	3	3
B	5	2
C	1	5
D	4	4
E	2	4

Thus, the minimum time taken is = 3+2+5+4+4=18 hours

Special cases in Assignment Problem

1. **Multiple Optimal Solution**
   We may have more than one alternative optimal solution for a given assignment problem. There may be two or more ways to cross out all zero elements in the final reduced matrix. This implies that there are more than required numbers of independent zero elements. In such situations, there will be multiple optimum solutions with the same total cost of assignment.

Example 3. A workshop put four mechanics to four different jobs. Find out how the jobs be allocated to each mechanic so as minimise man-hours needed.

Jobs

		1	**2**	**3**	**4**
	A	4	2	1	7
Mechanics	B	6	8	1	5
	C	5	3	4	6
	D	4	6	6	7

Solution:

Step 1: Find the First Reduce Matrix by selecting the smallest element of each row and subtract this smallest element from all the elements of that row (table 10).

Table 10: First reduced matrix (row operation)

Jobs

		1	**2**	**3**	**4**
	A	3	1	0	6
Mechanics	B	5	7	0	4
	C	2	0	1	3
	D	0	2	2	3

Step 2: Find the Second Reduced Matrix by selecting the minimum element from each column and subtract this element from all the elements of that column (table 11).

Table 11: Second reduced matrix (column operation)

Jobs

		1	**2**	**3**	**4**
	A	3	1	0	3
Mechanics	B	5	7	0	1
	C	2	0	1	0
	D	0	2	2	0

Step 3: Now examine each row consecutively for a single zero. Then assign (box) cell with zero and cross the remaining zeros in the respective columns and repeat the same procedure to obtain table 12.

Table 12

Jobs

		1	**2**	**3**	**4**
	A	3	1	[0]	3
Mechanics	B	5	7	~~0~~	1
	C	2	[0]	1	~~0~~
	D	[0]	2	2	~~0~~

Step 4: Draw minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix (table 13). Examine the elements that do not have a line through them. Choose the smallest of these elements and subtract this smallest from all the elements that do not have a line through them. Add this smallest element to every element that lies at the intersection of two lines. Then we obtain table 14 providing second feasible solution.

Table 13

Jobs

		1	2	3	4
	A	3	1	0	3
Mechanics	B	5	7	0	1
	C	2	0	1	0
	D	0	2	2	0

Table 14

Jobs

		1	2	3	4
	A	2	0	0	2
Mechanics	B	4	6	0	0
	C	2	0	2	0
	D	0	2	3	0

Step 5: Now repeat step 3 and make zero assignments as given in the table 15. Here an assignment problem can have more than one optimal solution. The Other solution is given in the table 16.

Table 15

Jobs

		1	2	3	4
	A	2	0	0	2
Mechanics	B	4	6	0	0
	C	2	0	2	0
	D	0	2	3	0

Optimal solution I

Mechanics	Jobs	Man-hours
A	2	2
B	3	1
C	4	6
D	1	4

Thus, the minimum man-hours needed is= 2+1+6+4=13 hrs

Table 16

Jobs

		1	2	3	4
	A	2	0	0	2
Mechanics	B	4	6	0	0
	C	2	0	2	0
	D	0	2	3	0

Optimal solution II

Mechanics	Jobs	Man-hours
A	3	1
B	4	5
C	2	3
D	1	4

Thus, the minimum man-hours needed is= 1+5+3+4=13 hrs

2. Maximization case in Assignment Problem

There may be situation when the assignment problem calls for maximization of profit, sales, revenues etc. Such problem can be solved by converting the given maximization problem into minimization problem by subtracting all the elements of the given matrix from the highest element. Thus, we get an opportunity loss matrix and we can apply Hungarian procedure to obtain optimal solution. Finally, we can find maximum profit for the given maximization assignment problem by restoring values of those cells where the assignment has been made.

Example 4: Four salesmen are assigned to four districts. Find the assignment pattern that maximizes the sales revenue.

Districts

		A	B	C	D
Salesmen	**1**	12	19	15	12
	2	16	18	17	16
	3	14	16	15	13
	4	14	12	17	14

Solution:

Step 1: Convert profit matrix (maximum sales table) into loss matrix (minimum sales table) by subtracting all elements of table from the largest element, then the resulting opportunity loss matrix is shown by table 17.

Table 17

Districts

		A	B	C	D
Salesmen	**1**	7	0	4	7
	2	3	1	2	3
	3	5	3	4	6
	4	5	7	2	5

Step 2: Now in loss matrix select the smallest element in each row and subtract this smallest element from all the elements of that row (table 18).

Table 18: (Row Operation)

Districts

		A	**B**	**C**	**D**
	1	7	0	4	7
Salesmen	**2**	2	0	1	2
	3	2	0	1	3
	4	3	5	0	3

Step 3: Now in loss matrix select the minimum element from each column and subtract this element from all the elements of that column (table 19).

Table 19: (Column Operation)

Districts

		A	**B**	**C**	**D**
	1	5	0	4	5
Salesmen	**2**	0	0	1	0
	3	0	0	1	1
	4	1	5	0	1

Step 4: Now examine each row consecutively for a single zero. Then assign (box) cell with zero and cross the remaining zeros in the respective columns and repeat the same procedure to obtain table 20.

Table 20

Districts

		A	**B**	**C**	**D**
	1	5	[0]	4	5
Salesmen	**2**	~~0~~	~~0~~	1	[0]
	3	[0]	~~0~~	1	1
	4	1	5	[0]	1

In the above table, there is one assignment in each row and each column. Thus optimal assignment for maximum sales is

Salesmen	Districts	Sales
1	B	19
2	D	16
3	A	14
4	C	17

Thus, maximum sales will be = 19+16+14+17= Rs 66

3. Unbalanced Assignment Problem

In the previous section, we assumed that the number of persons to be assigned and the number of jobs were same. Such kind of assignment problem is called as balanced assignment problem. Unbalanced Assignment problem is an assignment problem where the number of facilities is not equal to the number of jobs. To make unbalanced assignment problem, a balanced one, a dummy facility(s) or a dummy job(s) (as the case may be) is introduced with zero cost or time and then the problem is solved using the Hungarian Method. Here the jobs assigned