SRI JAI NARAIN MISHRA PG COLLEGE LUCKNOW

CO-ORDINATE GEOMETRY

CONFOCAL CONICS

Dr. Debangana Rajput Associate Professor
Department of Mathematics

BSc I Semester -2 Paper- 2 Unit-1

CONFOCAL CONICS

Definition: The conics, which have the common focus or foci are called confocal conics..

Equation of the system of confocal conics to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The given ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ....(1)$$

In this equation of the ellipse we know that the axes of the ellipse are taken as the co-ordinate axes and its centre as the origin. Since the confocal conics have the same foci(as that of ellipse) so will have the same centre and their axes in the same directiions.

In view of this concept the general equation of the conic confocal with the given ellipse will be

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \qquad ......(2)$$

The foci of the ellipses (1) and (2) are

$(\pm\sqrt{(a^2 - b^2)}, 0)$ and $(\pm\sqrt{(\alpha^2 - \beta^2)}, 0)$

By definition of confocal conics these co-ordinates of foci of (1) and (2) must be equal.

$\therefore \quad a^2 + b^2 = \alpha^2 - \beta^2 \quad$ Or $\quad \alpha^2 - a^2 = \beta^2 - b^2 = \lambda$ (say)

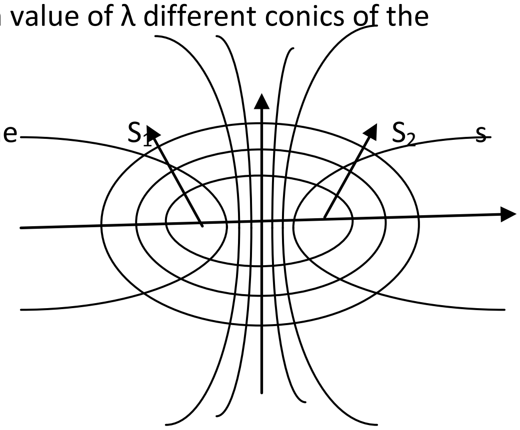
Or $\quad \alpha^2 = a^2 + \lambda$ and $\beta^2 = b^2 + \lambda$

Substituting the values of α^2 and β^2 in eqn (2) we get,

$$\boxed{\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1} \qquad ....(3)$$

Which is the required equation of the system of confocal conics to the given ellipse (1). Where λ is called the parameter of the confocal system. For each value of λ different conics of the confocal system can be obtained.

In adjacent figure all hyperbolas and ellipses have the same foci S_1 and S_2. So they are confocal conics.



Propositions on confocal conics:

(i) If through any point in the plane of an ellipse, there passess two confocal conics then one of them is an ellipse and the other is a hyperbola.

Proof: Let (h,k) be a point on the given ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...........(1)}$$

Let the equation of the conic confocal to (1) be

$$\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1 \qquad \text{........(2)}$$

If (2) passes through (h,k) then

$$\frac{h^2}{(a^2+\lambda)} + \frac{k^2}{(b^2+\lambda)} = 1 \qquad \text{.........(3)}$$

This is a quadratic equation in λ which gives two values of λ. Corresponding to each value of λ we get an equation representing confocal to (1).

Let $(b^2 + \lambda) = \mu$ then $(a^2 + \lambda) = (a^2 + \mu - b^2) = (a^2 - b^2) + \mu = a^2 e^2 + \mu$

Using $e = \sqrt{(1 - \frac{b^2}{a^2})} = \sqrt{(\frac{a^2 - b^2}{a^2})}$ or $ae = \sqrt{(a^2 - b^2)}$

Substituting $(b^2 + \lambda)$ and $(a^2 + \lambda)$ in (3) we obtain,

$$\frac{h^2}{(a^2 e^2+\mu)} + \frac{k^2}{\mu} = 1$$

$h^2 \mu + k^2 (a^2 e^2 + \mu) = \mu (a^2 e^2 + \mu)$

$h^2 \mu + k^2 (a^2 e^2 + \mu) = a^2 e^2 \mu + \mu^2$

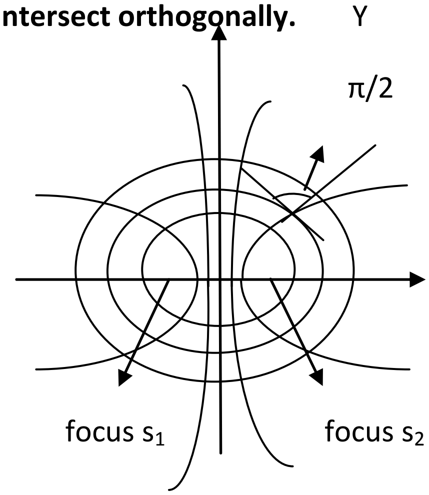
$\mu^2 + \mu (a^2 e^2 - h^2 - k^2) - a^2 e^2 k^2 = 0$ ....(4)

Here we can see that the product of roots of eqn (4) is $(- a^2 e^2 k^2)$, which shows that one value of $\mu > 0$ and other is < 0 i.e. $(b^2 + \lambda) > 0$ and $(b^2 + \lambda) < 0$.

Similarly for $(a^2 + \lambda) = \mu$ we get two positive roots of eqn (4) i.e. two positive values of $(a^2+ \lambda)$.

Taking positive values of $(a^2 + \lambda)$ and positive value of $(b^2 + \lambda)$ in eqn (2), we get, equation of confocal as an ellipse while taking positive values of $(a^2 + \lambda)$ and negative value of $(b^2 + \lambda)$ in eqn (2), we get a hyperbola. Hence, through a point two cofocals can be drawn, one of which is an ellipse and other is a hyperbola.

(ii) Confocal conics through any point in the plane of ellipse intersect orthogonally.



Let the equation of the conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the equation of the two confocals at the point (h,k)be

$$\frac{x^2}{(a^2+\lambda_1)} + \frac{y^2}{(b^2+\lambda_1)} = 1 \quad \text{..........(6)}$$

$$\frac{x^2}{(a^2+\lambda_2)} + \frac{y^2}{(b^2+\lambda_2)} = 1 \quad \text{..........(7)}$$

These confocals pass through M(h,k)

$$\frac{h^2}{(a^2+\lambda_1)} + \frac{k^2}{(b^2+\lambda_1)} = 1 \quad \text{...........(8)}$$

$$\frac{h^2}{(a^2+\lambda_2)} + \frac{k^2}{(b^2+\lambda_2)} = 1 \quad \text{...........(9)}$$

Subtracting (7) from (6),

$$h^2\left[\frac{1}{(a^2+\lambda_1)} - \frac{1}{(a^2+\lambda_2)}\right] + k^2\left[\frac{1}{(b^2+\lambda_1)} - \frac{1}{(b^2+\lambda_2)}\right] = 0$$

$$h^2\left[\frac{(\lambda_2-\lambda_1)}{((a^2+\lambda_1)(a^2+\lambda_2))}\right] + k^2\left[\frac{(\lambda_2-\lambda_1)}{((b^2+\lambda_1)(b^2+\lambda_2))}\right] = 0$$

Dividing it by $(\lambda_2 - \lambda_1)$ we get,

$$\left[\frac{h^2}{((a^2+\lambda_1)(a^2+\lambda_2))}\right] + \left[\frac{k^2}{((b^2+\lambda_1)(b^2+\lambda_2))}\right] = 0 \quad \text{..........(10)}$$

Now, the equations of tangents to (4) and (5) at point (h,k) are ,

$$\frac{xh}{(a^2+\lambda_1)} + \frac{yk}{(b^2+\lambda_1)} = 0 \quad \text{.........(11)}$$

$$\frac{xh}{(a^2+\lambda_2)} + \frac{yk}{(b^2+\lambda_2)} = 0 \quad \text{.........(12)}$$

The slope of the tangent (9) is

$$m_1 = -\left(\frac{h/(a^2+\lambda_1)}{k/(b^2+\lambda_1)}\right) = -\left(\frac{h(b^2+\lambda_1)}{k(a^2+\lambda_1)}\right)$$

Similarly, the slope of the tangent (10) is

$$m_2 = -\left(\frac{h(b^2+\lambda_2)}{k(a^2+\lambda_2)}\right)$$

The tangent lines given by (9) and (10) are perpendicular to each other if $m_1 \times m_2 = -1$

$$\therefore \quad -\left(\frac{h(b^2+\lambda_1)}{k(a^2+\lambda_1)}\right) \times -\left(\frac{h(b^2+\lambda_2)}{k(a^2+\lambda_2)}\right) = -1$$

$$\frac{h^2}{(a^2+\lambda_1)(a^2+\lambda_2)} + \frac{k^2}{(b^2+\lambda_1)(b^2+\lambda_2)} = 0$$

Which is true by virtue of (8). So two confocals intersect at right angles at the point (h,k).

(iii) One and only one member of system of confocals will touch a given straight line.

Let the system of confocals be given by,

$$\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$$

where λ is parameter.

Let the equation of the given line be

$$lx + my + n = 0 . \quad \text{or} \quad \frac{l}{n} + \frac{m}{n} + 1 = 0 \qquad \text{.........(13)}$$

First we find the condition that the given line is a tangent to the conic. Let (h,k) be any point on the confocal. The eqn of tangent to the confocal at this point is

$$\frac{xh}{(a^2+\lambda)} + \frac{yk}{(b^2+\lambda)} = 1 \qquad \text{........(14)}$$

Eqn (11) and (12) represent the same line. Therefore comparing them we get,

$$\frac{l/n}{h/(a^2+\lambda)} = \frac{m/n}{k/(b^2+\lambda)} = \frac{1}{1}$$

$$\text{Or } h = \frac{l}{n}(a^2+\lambda) \qquad \text{and} \qquad k = \frac{m}{n}(b^2+\lambda) \qquad \text{...........(15)}$$

Since (h,k) lies on the confocal so we have

$$\frac{h^2}{(a^2+\lambda)} + \frac{k^2}{(b^2+\lambda)} = 1$$

Substituting the values of h and k from eqn (13) we obtain,

$$\frac{\left(\frac{l}{n}(a^2+\lambda)\right)^2}{(a^2+\lambda)} + \frac{\left(\frac{m}{n}(b^2+\lambda)\right)^2}{(b^2+\lambda)} = 1$$

$$l^2(a^2+\lambda) + m^2(b^2+\lambda) = n^2 \qquad ....(16)$$

Which is the condition that the given line will be a tangent to the given confocal.

We see that the eqn (14) is linear in λ which gives only one value of λ. This implies that there is only one member of confocal system to touch the given line.

Note: If the equation of the line is taken as x cos α + y sin α = p then the above condition (14) takes the following form

$$(a^2+\lambda)\cos^2\alpha + (b^2+\lambda)\sin^2\alpha = p^2 \qquad ....(17)$$

On substituting the value of p from (17) the equation of tangent to the confocal $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$ *is given by*

$$x\cos\alpha + y\sin\alpha = \sqrt{(a^2+\lambda)\cos^2\alpha + (b^2+\lambda)\sin^2\alpha} \qquad .... (18)$$

(iv) The point of intersection of two perpendicular tangents one each of two given confocals lies on a circle.

Let the given two confocals be

$$\frac{x^2}{(a^2+\lambda_1)} + \frac{y^2}{(b^2+\lambda_1)} = 1$$

$$\frac{x^2}{(a^2+\lambda_2)} + \frac{y^2}{(b^2+\lambda_2)} = 1$$

The tangents to the these confocals are perpendicular to each other so using eqn (18) the equations of tangents to confocals are

$$x\cos\alpha + y\sin\alpha = \sqrt{(a^2+\lambda_1)\cos^2\alpha + (b^2+\lambda_1)\sin^2\alpha} \qquad ....(19)$$

$$x\sin\alpha - y\cos\alpha = \sqrt{(a^2+\lambda_2)\sin^2\alpha + (b^2+\lambda_2)\cos^2\alpha} \qquad ....(20)$$ [It is perpendicular to eqn (19)]

Squaring and adding these two, we get

$$(x\cos\alpha + y\sin\alpha)^2 + (x\sin\alpha - y\cos\alpha)^2 = \{(a^2+\lambda_1)\cos^2\alpha + (b^2+\lambda_1)\sin^2\alpha\} +$$

$$\{(a^2 + \lambda_2) \sin^2\alpha + (b^2 + \lambda_2)\cos^2\alpha\}$$

$x^2 \cos^2 \alpha + y^2 \sin^2 \alpha + x^2 \sin^2 \alpha + y^2 \cos^2 \alpha = (a^2 + b^2) + \lambda_1 + \lambda_2.$

$x^2 + y^2 = (a^2 + b^2) + \lambda_1 + \lambda_2$

which is the equation of a circle.

(v) The locus of the pole of a given straight line with respect to a system of confocals is a straight line.

Let the given line be $lx + my = 1$ .... (21)

and the system of confocals be $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1.$

Let (h,k) be the pole of the given line $lx + my = 1$ with respect to the given system of confocals then the polar of (h,k) with respect to the given confocals is

$\frac{xh}{(a^2+\lambda)} + \frac{yk}{(b^2+\lambda)} = 1.$ ....(22) [T=0 is the eq of polar]

Eqns (21) and (22) represent the same line,therefore comparing them

$\frac{l}{h/(a^2+\lambda)} = \frac{m}{k/(b^2+\lambda)} = 1$ or $\frac{h}{l} = (a^2 + \lambda)$, $\frac{k}{m} = (b^2 + \lambda)$

Or $\lambda = \frac{h}{l} - a^2 = \frac{k}{m} - b^2$ or $\frac{h}{l} - \frac{k}{m} = a^2 - b^2$

Hence the locus of (h,k) is

or $\frac{x}{l} - \frac{y}{m} = a^2 - b^2$

which is a straight line.

(vi) The difference of the squares of perpendiculars drawn from the centre on any two parallel tangents to two given confocal conics is constant.

Let the equation of two confocal conics be

$$\frac{x^2}{(a^2+\lambda_1)} + \frac{y^2}{(b^2+\lambda_1)} = 1$$

$$\frac{x^2}{(a^2+\lambda_2)} + \frac{y^2}{(b^2+\lambda_2)} = 1$$

Let the parallel lines be

$x \cos\alpha + y \sin\alpha = p_1$

and $x \cos\alpha + y \sin\alpha = p_2$

touch the confocal conics. So by condition of tangency eqn (17) we have

$(a^2 + \lambda_1)\cos^2\alpha + (b^2 + \lambda_1)\sin^2\alpha = {p_1}^2$ ....(23)

$(a^2 + \lambda_2)\cos^2\alpha + (b^2 + \lambda_2)\sin^2\alpha = {p_2}^2$ ....(24)

Subtracting (24) from (23)we get,

${p_1}^2 - {p_2}^2 = \lambda_1 - \lambda_2$, which is a constant.

Examples on confocal conics

(1) Find the equation of conic which is confocal to the conic $x^2 + 9y^2 = 9$ and pass through the point (3,1).

Solution: Given: $x^2 + 9y^2 = 9$ or $\frac{x^2}{9} + \frac{y^2}{1} = 1$ ......(1)

Let the equation of conic confocal to (1) be,

$\frac{x^2}{(9+\lambda)} + \frac{y^2}{(1+\lambda)} = 1$ ....(2)

The equation (2) passes through the point (3,1),

$\therefore \frac{9}{(9+\lambda)} + \frac{1}{(1+\lambda)} = 1$ or $9(1+\lambda) + 9 + \lambda = (9+\lambda)(1+\lambda) = 9 + 10\lambda + \lambda^2$

$$\lambda^2 = 9 \quad \text{or } \lambda = \pm 3$$

Therefore, the required confocals are ,

$\frac{x^2}{12} + \frac{y^2}{4} = 1$ taking $\lambda = 3$ (an ellipse)

$\frac{x^2}{6} - \frac{y^2}{2} = 1$ taking $\lambda = -3$ (a hyperbola)

(2) Find the locus of the mid-point of the chords of the confocal $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$, which touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: The given confocal is $\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$ ....(1)

Let (h,k) be the mid-point of the chords of (1), then the equation of the chord is given by

$$\frac{xh}{(a^2+\lambda)} + \frac{yk}{(b^2+\lambda)} = \frac{h^2}{(a^2+\lambda)} + \frac{k^2}{(b^2+\lambda)} \quad ....(2)$$

[Equation of chord with given middle point is T = S_1 where T and S_1 has their usual meaning]

The given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(3)

Given that (2) touches (3), therefore we have

$$\left(\frac{h}{(a^2+\lambda)}\right)^2 a^2 + \left(\frac{k}{(b^2+\lambda)}\right)^2 b^2 = \left[\frac{h^2}{(a^2+\lambda)} + \frac{k^2}{(b^2+\lambda)}\right]^2$$

[Using the condition $a^2 m^2 + b^2 = c^2$, which is the required condition for a line y = mx + c to be the tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$]

$$\frac{h^2a^2}{(a^2 + \lambda)^2} + \frac{k^2b^2}{(b^2 + \lambda)^2} = \left[\frac{h^2}{(a^2 + \lambda)} + \frac{k^2}{(b^2 + \lambda)}\right]^2$$

Thus the locus of (h,k) is

$$\frac{x^2a^2}{(a^2 + \lambda)^2} + \frac{y^2b^2}{(b^2 + \lambda)^2} = \left[\frac{x^2}{(a^2 + \lambda)} + \frac{y^2}{(b^2 + \lambda)}\right]^2$$