SUBJECT: MATHEMATICS

PAPER-II, UNIT-III

SIMPLE HARMONIC MOTION
SEMESTER-IV

DR. JITENDRA AWASTHI
DEPARTMENT OF MATHEMATICS
S. J. N. M. P. G. COLLEGE, LUCKNOW

Simple Harmonic Motion (S.H.M) is an interesting special type of motion in nature, having forward and backward oscillation (or) to and fro oscillation about a fixed point. The fixed point is known as the mean position or equilibrium position. When the oscillation is very small we prove the motion is simple harmonic. In this section we study about the resultant of two S.H.M"S of the same period in the same straight line and in two perpendicular lines. Also we find the periodic time of oscillation of a simple pendulum.

Examples

Small oscillation of a cradle, simple pendulum, seconds pendulum, simple equivalent pendulum, transverse vibrations of a plucked violin string etc.

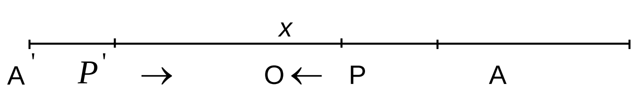
Hooke's law

Tension of an elastic string or spring is directly proportional to its extended length and indirectly proportional to its natural length.

1.1 Simple Harmonic Motion in a straight line

Definition

When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.



Let O be a fixed point on the straight line A^1 OA on which a particle is having simple harmonic motion. Take O as the origin and OA as the X axis. Let P be the position of the particle at time t such that OP = x.

The magnitude of the acceleration at P in $-\mu x\ where\ \mu\ is\ a$ constant.

The acceleration at P in the positive direction of the X axis is μx

Hence the equation of motion of P is

$$\frac{d^2x}{dt^2} = -\mu x \quad \dots\dots\dots(1)$$

Equation (1) is the fundamental differential equation representing a S.H.M.

If v is the velocity of the particle at time t (1) can be written as

$$\boldsymbol{v\frac{dv}{dx} = -\mu x}$$

i.e. $vdv = -\mu x\, dx$……...(2)

$$or\ \frac{v^2}{2} = -\frac{x^2}{2} + c \ldots\ldots\ldots\ldots\ldots.(3)$$

Initially let the particle starts from rest at the point A where OA = a

Hence when x=a, v = 0 = $\frac{dx}{dt}$

Putting these in (3), 0 = $-\frac{\mu a^2}{2}$ +c or c = $\frac{\mu a^2}{2}$

$\therefore v^2 = -\mu\ x^2 + \mu a^2 = \mu\left(a^2 - x^2\right)$

$\therefore$ v = $\pm\ \sqrt{\mu\ \left(a^2 - x^2\right)}$**………**(4)

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence we take the negative sign in (4),

$\frac{dx}{dt}$ **= v =** $-\sqrt{\mu\left(a^2 - x^2\right)}$ **……..(5)**

$$-\frac{dx}{\sqrt{\left(a^2 - x^2\right)}} = \sqrt{\mu}\ dt$$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t + A$

Initially when t = 0, x = a, $\cos^{-1} 1 = 0 + A \Rightarrow$ A=0

$\therefore\ \cos^{-1}\frac{x}{a} = \sqrt{\mu}\ t$ or x = a cos $\sqrt{\mu}\ t$ ....(6)

To get the time from A to A^1, put x = $-a$ in (6)

We have cos $\sqrt{\mu}\,t = -1 = \cos\pi$ t = $\frac{\pi}{\sqrt{\mu}}$

$\therefore$ The time from A to A' and back = $\dfrac{2\pi}{\sqrt{\mu}}$.

Equation (6) can be written as

x = a cos $\sqrt{\mu}$ t = a cos $(\sqrt{\mu}\ t + 2\pi)$ = a cos$(\sqrt{\mu}\ t + 4\pi)$ etc

= a cos $\sqrt{\mu}\left(t + \dfrac{2\pi}{\sqrt{\mu}}\right)$ = a cos $\sqrt{\mu}\left(t + \dfrac{4\pi}{\sqrt{\mu}}\right)$ etc.

Differentiating (6),

$\dfrac{dx}{dt} = -a\sqrt{\mu}$. sin $\sqrt{\mu}$ t

= $-\ a\sqrt{\mu}$ sin $(\sqrt{\mu}\ t + 2\pi)$ = $-\ a\ \sqrt{\mu}$ sin $(\sqrt{\mu}\ t + 4\pi)$ etc.

= $-\ a\sqrt{\mu}$ sin $\sqrt{\mu}\ (t + \dfrac{2\pi}{\sqrt{\mu}})$ = $-\ a\sqrt{\mu}$ sin $\sqrt{\mu}\ (t + \dfrac{4\pi}{\sqrt{\mu}})$ etc.

The values of $\dfrac{dx}{dt}$ are the same if t is increased by $\dfrac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\dfrac{2\pi}{\sqrt{\mu}}$. Hence after a time $\dfrac{2\pi}{\sqrt{\mu}}$ the particle is again at the same point moving with the same velocity in the same direction. Hence the particle has the period $\dfrac{2\pi}{\sqrt{\mu}}$.

T = $\dfrac{2\pi}{\sqrt{\mu}}$; frequency = $\dfrac{1}{T}$ = $\dfrac{2\pi}{\sqrt{\mu}}$

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Amplitude = OA = OA' = a.

The periodic time = $\dfrac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Deductions : 1) Maximum acceleration $= \mu.a$ $= \mu$. (amplitude)

2) Since v = $\sqrt{\mu(a^2 - x^2)}$, the greatest value of v is at x = 0 and its

Maximum velocity = a $\sqrt{\mu}$ = $\sqrt{\mu}$. (amplitude) at the centre

General solution of the S.H.M. equation

The S.H.M. equation is $\frac{d^2x}{dt^2} = -\mu x$

i.e. $\frac{d^2x}{dt^2} + \mu x = 0$ …….(1)

(1) is a differential equation of the second order with constant coefficients. Its general solution is of the form

x = A cos $\sqrt{\mu}$ t + B sin $\sqrt{\mu}$ t ....(2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

x = C cos ($\sqrt{\mu}$ t + ε)…. (3) and x = D sin ($\sqrt{\mu}$ t + α) ………(4)

- If the solution of the S.H.M. equation is x = a cos ($\sqrt{\mu}$ t + ε), the quantity ε is called the **epoch**.

Definition

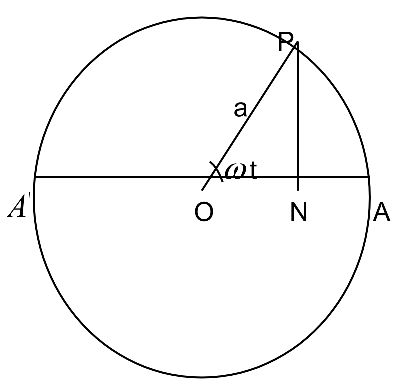
If two simple harmonic motions of the same period can be represented by

$x_1 = a_1 \cos(\sqrt{\mu}\, t + \varepsilon_1)$ and $x_2 = a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2)$

- The difference in phase = $\frac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$
- If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.
- If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phase.

1.2 Geometrical Representation of S.H.M

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with S.H.M.



Let AA' be the diameter of the circle with centre O and P be the position of the particle at time t secs .Let N be the foot of the perpendicular drawn from P on the diameter AA'.
P moves along the circumference of the circle with uniform speed and describes equal arcs in equal times. Let ω be the angular velocity. $\therefore \angle AOP = \omega t$

If ON = x, Op = a, then, $x = a\cos(\omega t)$.......................... (1)

$$\frac{dx}{dt} = -a\omega \sin(\omega t) \quad \text{..................(2)}$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos(\omega t) = -\omega^2 x \quad \text{................. (3)}$$

(3) shows that the motion of N is simple harmonic. When P moves along the circumference of the circle starting from A, N oscillates from A to A' and A' to A.

Periodic time of P = Periodic time of N= $2\pi/\omega$

(along the circle) (along the diameter)

Example 1

A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are $x_1, x_2, x_3.$ Prove that the period of oscillation is $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1+x_3}{2x_2}\right)}$

Solution:

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that x = a cos $\sqrt{\mu}$ t ….. (1)

Let $x_1, x_2, x_3.$ be the displacements at three consecutive seconds $t_1,\ t_1+1,\ t_1+2$.

Then $x_1 = a\cos\sqrt{\mu}\, t_1$...(2)

$x_2 = a\cos\sqrt{\mu}(t_1+1) = a\cos\left(\sqrt{\mu}t_1+\sqrt{\mu}\right)$…….(3)

$x_3 = a\cos\sqrt{\mu}(t_1+2) = a\cos\left(\sqrt{\mu}t_1+2\sqrt{\mu}\right)$ …….(4)

$\therefore\ x_1 + x_3 = a\,[\cos\left(\sqrt{\mu}\ t_1 + 2\sqrt{\mu}\right) + \cos\left(\sqrt{\mu}\ t_1\right)]$

$= a.2\cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} + \sqrt{\mu}t_1}{2}\ .\cos\dfrac{\sqrt{\mu}t_1 + 2\sqrt{\mu} - \sqrt{\mu}t_1}{2}$

$= 2\,a\cos\left(\sqrt{\mu}\ t_1 + \sqrt{\mu}\right).\cos\sqrt{\mu}\ = 2x_2\,.\cos\sqrt{\mu}$

$\therefore \dfrac{x_1 + x_3}{2x_2} = \cos\sqrt{\mu}\ ,\qquad \sqrt{\mu} = \cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)$

$\text{Period} = \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$

Example 2

If the displacement of a moving point at any time be given by an equation of the form $x = a\cos\omega t + b\sin\omega t$, show that the motion is a simple harmonic motion.

If a = 3, b=4, $\omega = 2$ determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

Solution:

Given $x = a\cos\omega t + b\sin\omega t$ ....(1)

Differentiating (1) with respect to t,

$\dfrac{dx}{dt} = -a\omega\sin\omega t + b\omega\cos\omega t$ ……………………(2)

$\dfrac{d^2x}{dt^2} = -\omega^2\cos\omega t - b\,\omega^2\sin\omega t$

$= -\omega^2(a\cos\omega t + b\sin\omega t) = -\omega^2 x$ ….(3)

$\therefore$ The motion is simple harmonic.

The constant μ of the S,H.M. = ω^2.

$\therefore$ Period = $\dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ secs.

Amplitude is the greatest value of x.

When x is maximum, $\frac{dx}{dt} = 0$.

$-a\omega \sin \omega t + b\omega \cos \omega t$ = 0 i.e. a sin ωt = b cos ω t or tan ω t = $\frac{b}{a} = \frac{4}{3}$

When tan ω t = $\frac{4}{3}$, sin ω t = $\frac{4}{5}$ and cos ω t = $\frac{3}{5}$

Greatest value of x = a $\times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a + 4b}{5} = \frac{3.3 + 4.4}{5} = 5$

Hence amplitude = 5.

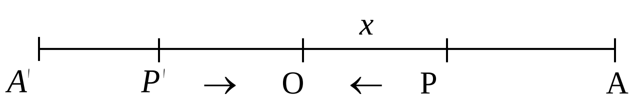
Max. acceleration = μ . Amplitude = 4 x 5 = 20

Max. velocity = $\sqrt{\mu}$. Amplitude = 2 x 5 =10

Example 3

Show that the energy of a system executing S.H.M. is proportional to the square of the amplitude and of the frequency.

Solution:



The acceleration at a distance x from O = μ x.

Force = mass ×acceleration = m μx

If the particle is given displacement dx from P,

work done against the force = m $\mu\, x.$dx

Total work done in displacing the particle to a distance x

$$= \int_0^x m \mu x dx = m \mu \frac{x^2}{2} \quad \text{.........(1)}$$

Work done = potential energy at P.

If v is the velocity at P. we know that v $^2 = \mu\left(a^2 - x^2\right)$,

$\therefore$ Kinetic energy at P = $\frac{1}{2}$ mv^2= $\frac{1}{2} m\mu\left(a^2 - x^2\right)$ ……….. (2)

The total energy at P = Potential energy + Kinetic energy

$$= \frac{m\mu x^2}{2} + \frac{m\mu}{2}\left(a^2 - x^2\right) \qquad \ldots\ldots\ldots (3)$$

Total energy at P $\alpha\, a^2$

If n is the frequency, we know that

$$n = \frac{1}{Period} = \frac{1}{\left(\frac{2\pi}{\sqrt{\mu}}\right)} = \frac{\sqrt{\mu}}{2\pi}$$

$\therefore\ \sqrt{\mu} = 2\pi$ n or $\mu = 4\pi^2 n^2$

Total energy = $\frac{1}{2} m.\, 4\pi^2 n^2 a^2 = 2\pi^2 m a^2 n^2 \quad \alpha\, n^2$

Example 4

A mass of 1 gm. Vibrates through a millimeter on each side of the midpoint of its path 256 times per sec; if the motion be simple harmonic, find the maximum velocity,

Solution:

Maximum velocity v = $\sqrt{\mu}$. a

Given, frequency = $\frac{1}{T}$ = 256 = $\frac{\sqrt{\mu}}{2\pi}$.

$\therefore\ \sqrt{\mu}$ = 2 × 256 × π.

Given, amplitude = a = 1 millimeter = 1 × 10 $^{-1}$ c.m.

$\therefore$ Maximum velocity, V = 2 × 256 × π × $\frac{1}{10}$ = $\frac{256\ \pi}{5}$ cm/ sec

Example 5

A body moving with simple harmonic motion has an amplitude „a" and period T. Show that the velocity v at a distance x from the mean position is given by $v^2 T^2 = 4\pi^2 \left(a^2 - x^2\right)$

Solution:

We know, $v^2 = \mu\left(a^2 - x^2\right)$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

$$\therefore v^2 = \frac{4\pi^2}{T^2}\left(a^2 - x^2\right)$$

$$\therefore v^2T^2 = 4\pi^2\left(a^2 - x^2\right)$$

Example 6

A particle, moving in S.H.M. has amplitude 8 cm. If its maximum acceleration is 2cm/sec ^{2}, find (i) its period (ii) maximum velocity and (iii) its velocity when it is 3 cm. from the extreme position

Solution:

Maximum acceleration = 2 cm/ sec ^{2} = $\mu.a. = \mu \times 8$.

$$\therefore \mu = \frac{2}{8} = \frac{1}{4},$$

Period T = $\dfrac{2\pi}{\sqrt{\mu}}$ = 2 $\pi \times \dfrac{1}{\sqrt{\frac{1}{4}}} = 4\pi$ secs.

Max. velocity = $\sqrt{\mu}.$ a = $\dfrac{1}{2} \times 8 = 4cm$ / sec.

When the particle is 3 cm from the extreme position, x = 5 cm.

$$\therefore \text{ velocity}^2 = v^2 = \mu\left(a^2 - x^2\right) = \frac{1}{4}\left(64 - 25\right) = \frac{39}{4}.$$

$$\therefore v = \tfrac{1}{2}\sqrt{39} \text{ cm / sec.}$$

Example 7

If the distance x of a point moving on a straight line measured from a fixed origin on it and velocity v are connected by the relation $4v^2 = 25 - x^2$, show that the motion is simple harmonic..

Solution:

Given, $4v^2 = 25 - x^2$ .... (1)

Differentiating, $8v. \dfrac{dv}{dt} = -2x. \dfrac{dx}{dt}$

$$\therefore \frac{dv}{dt} = -\frac{1}{4}.x.$$

$$\frac{d^2x}{dt^2} = -\frac{1}{4}.x.$$

Hence the motion is a S.H.M.

1.3Composition of two simple Harmonic Motions of the same period and in the same straight line

Since the period same, the two separate simple harmonic motions are represented by the same differential equation $\frac{d^2x}{dt^2} = -\mu x$

Let x_1 and x_2 be the displacements for the separate motions.

$x_1 = a_1 \cos\left(\sqrt{\mu}\, t + \varepsilon_1\right)$, a_1 - amplitude

$x_2 = a_2 \cos\left(\sqrt{\mu}\, t + \varepsilon_2\right)$, a_2 – amplitude

Let x be their resultant displacement, then $x = x_1 + x_2$

ie) $x = a_1\cos\left(\sqrt{\mu}t + \varepsilon_1\right) + a_2\cos\left(\sqrt{\mu}t + \varepsilon_2\right)$

$= a_1\left[\cos\sqrt{\mu}t.\cos\varepsilon_1 - \sin\sqrt{\mu}t.\sin\varepsilon_1\right] + a_2\left[\cos\sqrt{\mu}t.\cos\varepsilon_2 - \sin\sqrt{\mu}t.\sin\varepsilon_2\right]$

$= \cos\sqrt{\mu}t\left(a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2\right) - \sin\sqrt{\mu}t\left(a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2\right)$

$= \cos\sqrt{\mu}t.A\cos\varepsilon - \sin\sqrt{\mu}t.A\sin\varepsilon$ ……………... (1)

where $A\cos\varepsilon = a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2$ ……………..(2)

$A\sin\varepsilon = a_1\sin\varepsilon_1 + a_2\sin\varepsilon_2$ …………… (3)

Squaring (2) and (3) and adding,

$A^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\left(\varepsilon_1 - \varepsilon_2\right)$ ………..(4)

Dividing (3) by (2), tan $\varepsilon = \dfrac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2}$ ……..(5)

Now (1) becomes x = A . $\left(\cos\sqrt{\mu}t\cos\varepsilon - \sin\sqrt{\mu}t\sin\varepsilon\right)$

= A . cos $\left(\sqrt{\mu}t + \varepsilon\right)$ ……… (6)

The resultant displacement given by (6) also represents a simple harmonic motion of the same period as the individual motions.

Example 8

Two simple harmonic motions in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6, find the amplitude and phase of the resulting motion

Solution:

Let the two S.H.M. in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ be,

$x_1 = a_1.\cos\sqrt{\mu}$ t ....(1)

$x_2 = a_2\left(\cos\sqrt{\mu}\ t + \varepsilon\right)$ **......** (2)

given, A cos ε = 4 = a_1 , A sin ε = 6 = a_2

$\therefore$ Amplitude of the resultant motion A $= \sqrt{(A\cos\varepsilon)^2 + (ASin\varepsilon)^2}$

$= \sqrt{16+36} = \sqrt{52}$

$$\boxed{A = 2\sqrt{13}}$$

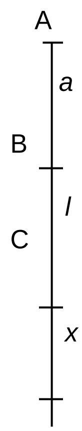
$$\tan\varepsilon = \frac{A\ Sin\varepsilon}{A\ Cos\varepsilon} = \frac{6}{4} = \frac{3}{2}$$

$$\boxed{\varepsilon = \tan^{-1}\left(\frac{3}{2}\right)}$$

which is the phase of the resulting motion.

1.4 Motion of a particle suspended by a spiral spring

A particle is suspended from a fixed point by a spiral spring of natural length a and modulus λ. If it is displaced slightly in the vertical direction, discuss the subsequent motions



Let AB = a, natural length of the spring which is fixed at A. Let m be the mass of the particle connected at B, which pulls the spring and comes to rest at C such that the increased length BC = *l*. At C, the mass „m" is in equilibrium. Hence the downward force mg and the upward force T must be equal at C. ie) T = mg

But, by Hooke"s law, T = $\frac{\lambda l}{a}$

$$\therefore \frac{\lambda l}{a} = mg \text{..} (1)$$

Let the particle be slightly displaced vertically downwards through a distance and then released. It will begin to move upwards. Let P be the subsequent position of the particle so that CP= x

The forces acting at P are the weight and the upward tension.

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = \text{Resultant downward force} = mg - \text{Tension at P.}$$

$$= mg - \frac{\lambda}{a}(AP-AB)$$

$$= mg - \frac{\lambda}{a}(BP) = mg - \frac{\lambda}{a}(l + x)$$

$$= -\frac{\lambda x}{a} \quad [\text{as } mg = \frac{\lambda l}{a}] \quad \text{by (1)}$$

$$\text{i.e. } \frac{d^2x}{dt^2} = -\frac{\lambda}{a\,m}x \quad \text{..... (2)}$$

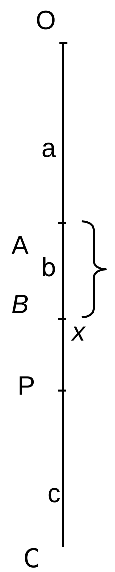
Equation (2) represents a S.H.M.

$$\text{Period} = \frac{2\pi}{\sqrt{\frac{\lambda}{am}}} = 2\pi\sqrt{\frac{am}{\lambda}}$$

Example 8

Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off. Show that the distance of M from the upper end of the string at time t is a+b+ c cos $\sqrt{\frac{g}{b}}$ t, where a is the unstretched length of the string, and b and c are the distances by which it would be stretched when supporting M and M', respectively.

Solution



Let OA = a be the natural length of the elastic string, which is fixed at O. When the string supports M,

Mg = upward Tension.

By Hooke's law,

upward Tension at B = $\frac{\lambda b}{a}$

$\therefore\ Mg = \frac{\lambda b}{a}$ ……………….. (1)

When the string supports M^1,

M^1g = upward Tension at C = $\frac{\lambda c}{a}$

ie) M^1g= $\frac{\lambda c}{a}$ …………………….. (2)

(1) + (2) $\Rightarrow M + M' = \frac{\lambda}{a}(b+c)$

ie) At C, M + M' is in equilibrium.

When M' falls off, M will move towards B.

Let P be the position of M at time t seconds such that BP = x

Forces acting at P are,

(i) Weight Mg ii) Upward tension

$\therefore$ At P, equation of motion of M is $M.\frac{d^2x}{dt^2}$ = resultant downward force.

$$= Mg - \frac{\lambda}{a}(OP - OA)$$

$$= Mg - \frac{\lambda}{a}(AP)$$

$$= Mg - \frac{\lambda}{a}(b + x)$$

$$= Mg - \frac{\lambda b}{a} - \frac{\lambda}{a}x$$

$$= -\frac{\lambda}{a}x \qquad \text{by (1)}$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{aM}.\ x$$

$\therefore$ The motion of M at P is simple harmonic

Amplitude = BC = c

$\therefore$ Displacement = $x = c.\cos\sqrt{\frac{\lambda}{aM}}\ t$

$$= c.\cos\sqrt{\frac{g}{b}}.\ t \qquad \text{by (1)}$$

$\therefore$ Distance of M from O at time t = OP = OA + AB + BP

= a + b + x

= a + b + c. cos $\sqrt{\frac{g}{b}}.\ t$