

**SUBJECT: MATHEMATICS**

**PAPER-II, UNIT-III**

**VELOCITIES AND ACCELERATIONS IN POLAR CURVE**

**SEMESTER-IV**

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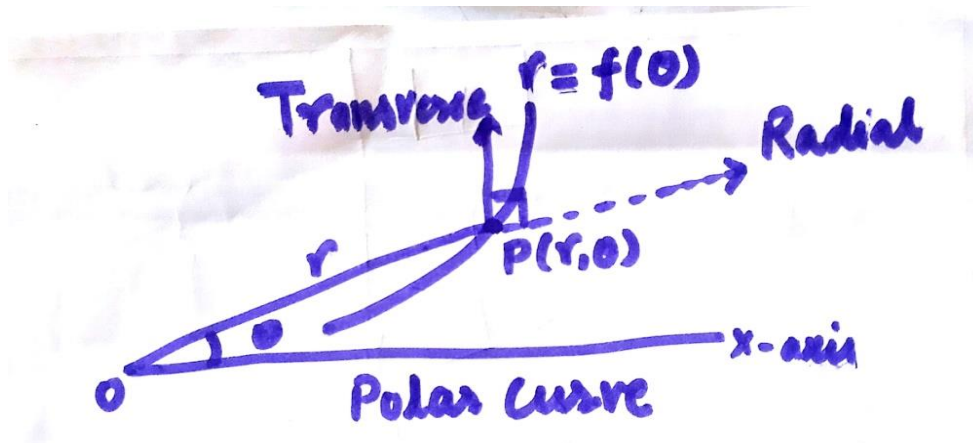
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In this chapter we study components of velocities and accelerations in two mutually perpendicular directions in a Polar curve.

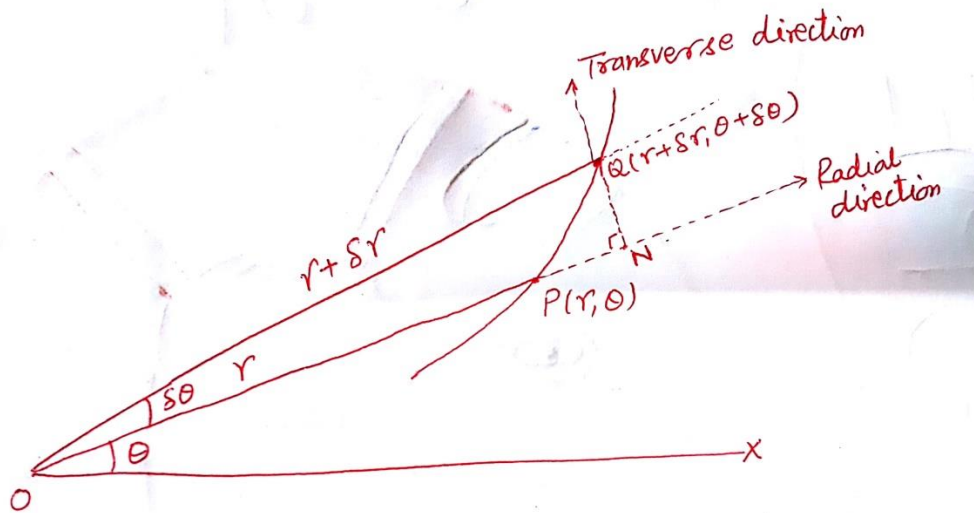
**POLAR CURVE:**

In a polar curve  $r = f(\theta)$ , there are two perpendicular directions, Radial and Transverse as shown in the figure.



**VELOCITY AND ACCELERATION IN POLAR CO- ORDINATES (RADIAL AND TRANSVERSE VELOCITIES):**

Consider a particle moves in a plane curve. Let  $P(r, \theta)$  be its position in time  $t$  and  $Q(r + \delta r, \theta + \delta \theta)$  be its position in time  $(t + \delta t)$ . Take 'O' as the pole and OX- as initial line. Velocity along the radius vector OP in the direction of  $r$  increasing is called the radial velocity and the velocity in the direction perpendicular to OP in the direction of  $\theta$  increasing is called the transverse velocity.



So Radial Velocity at P =  $\lim_{\delta t \rightarrow 0} \frac{\text{displacement along } OP \text{ in time } \delta t}{\delta t}$

$$= \lim_{\delta t \rightarrow 0} \frac{PN}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{ON - OP}{\delta t}$$

$$= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \cos \delta \theta - r}{\delta t}$$

$$= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \left[ 1 - \frac{(\delta \theta)^2}{2!} + \dots \dots \right] - r}{\delta t}$$

$$= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r)(1) - r}{\delta t}$$

(neglecting higher powers of  $\delta \theta$ )

$$= \lim_{\delta t \rightarrow 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} = \dot{r}$$

So, Radial velocity =  $\dot{r}$

Also, Transverse Velocity at P

$$= \lim_{\delta t \rightarrow 0} \frac{\text{displacement perpendicular } OP \text{ in time } \delta t}{\delta t}$$

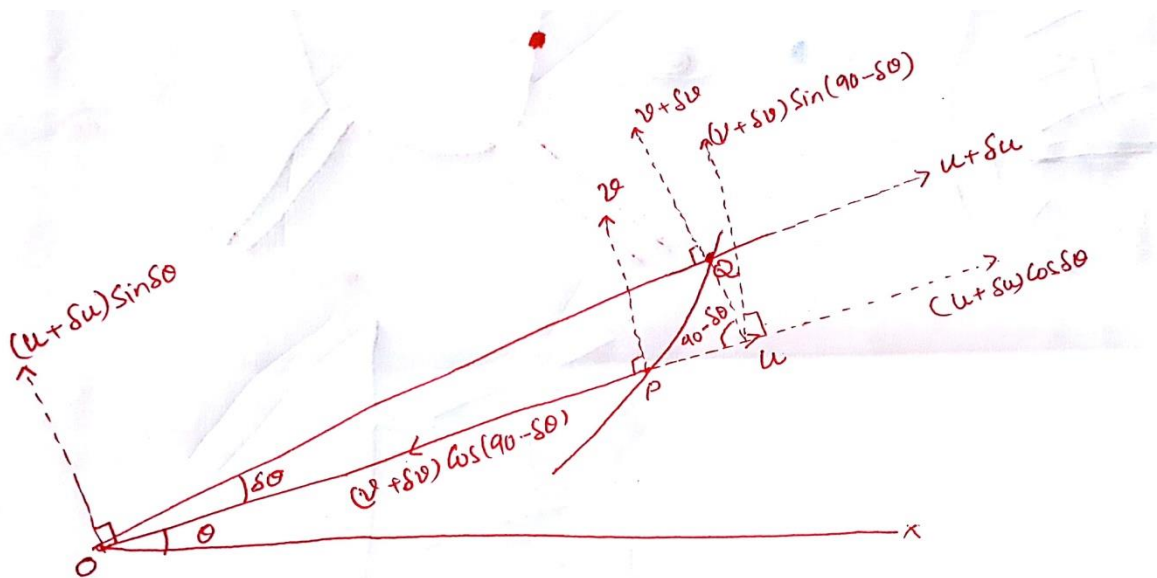
$$= \lim_{\delta t \rightarrow 0} \frac{QN}{\delta t}$$

$$\begin{aligned}
&= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \sin \delta \theta}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r) \left[ \delta \theta - \frac{(\delta \theta)^3}{3!} + \dots \right]}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{(r + \delta r)(\delta \theta)}{\delta t} \\
&\quad \text{(neglecting higher powers of } \delta \theta \text{)} \\
&= \lim_{\delta t \rightarrow 0} r \frac{\delta \theta}{\delta t} = r \frac{d\theta}{dt} = r\dot{\theta}
\end{aligned}$$

So, Transverse velocity =  $r\dot{\theta}$

#### RADIAL AND TRANSVERSE ACCELERATIONS:

Let  $u, v$  be the radial and transverse velocities at  $(r, \theta)$  and  $(u + \delta u), (v + \delta v)$  radial and transverse velocities at  $Q (r + \delta r, \theta + \delta \theta)$ .



So Radial acceleration at P =  $\lim_{\delta t \rightarrow 0} \frac{\text{change in velocity along OP in time } \delta t}{\delta t}$

$$\begin{aligned}
&= \lim_{\delta t \rightarrow 0} \frac{(u+\delta u) \cos \delta \theta - (v+\delta v) \cos(90^\circ - \delta \theta) - u}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{(u+\delta u)(1) - (v+\delta v)(\delta \theta) - u}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{\delta u - v \delta \theta}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{\delta u}{\delta t} - \lim_{\delta t \rightarrow 0} v \frac{\delta \theta}{\delta t} \\
&= \frac{du}{dt} - v \frac{d\theta}{dt} \\
&= \frac{d}{dt} \left( \frac{dr}{dt} \right) - \left( r \frac{d\theta}{dt} \right) \frac{d\theta}{dt} \text{ (as } u = \frac{dr}{dt}, v = r \frac{d\theta}{dt} \text{)}
\end{aligned}$$

So, **Radial acceleration**  $= \frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2$

Now, Transverse acceleration at P

$$\begin{aligned}
&= \lim_{\delta t \rightarrow 0} \frac{\text{change in velocity perpendicular to } OP \text{ in time } \delta t}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{(u+\delta u) \sin \delta \theta + (v+\delta v) \sin(90^\circ - \delta \theta) - v}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{(u+\delta u)(\delta \theta) + (v+\delta v)(1) - v}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} \frac{u(\delta \theta) + \delta v}{\delta t} \\
&= \lim_{\delta t \rightarrow 0} u \frac{\delta \theta}{\delta t} + \lim_{\delta t \rightarrow 0} \frac{\delta v}{\delta t} = u \frac{d\theta}{dt} + \frac{dv}{dt}
\end{aligned}$$

$$= \left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\right) \text{ (as } u = \frac{dr}{dt}, v = r\frac{d\theta}{dt}\text{)}$$

$$= 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

So, Transverse **acceleration** =  $\frac{1}{r}\frac{d}{dt}(r^2\ddot{\theta})$

	Velocities and Accelerations	Magnitude
1	Radial Component of velocity	$\dot{r}$
2	Transverse Component of velocity	$r\dot{\theta}$
3	Radial component of acceleration	$\ddot{r} - r(\dot{\theta})^2$
4	Transverse component of acceleration	$\frac{1}{r}\frac{d}{dt}(r^2\ddot{\theta})$

**Example:** If a point moves so that its radial velocity is k times its transverse velocity, then show that its path is an equiangular spiral.

**Solution:**

Given, radial velocity =  $k \times$  transverse velocity

$$\Rightarrow \dot{r} = kr\dot{\theta}$$

$$\text{i.e. } \frac{dr}{dt} = kr\frac{d\theta}{dt} \Rightarrow \frac{dr}{r} = kd\theta$$

Integrating,  $\log r = k\theta + \log A$ , A = constant

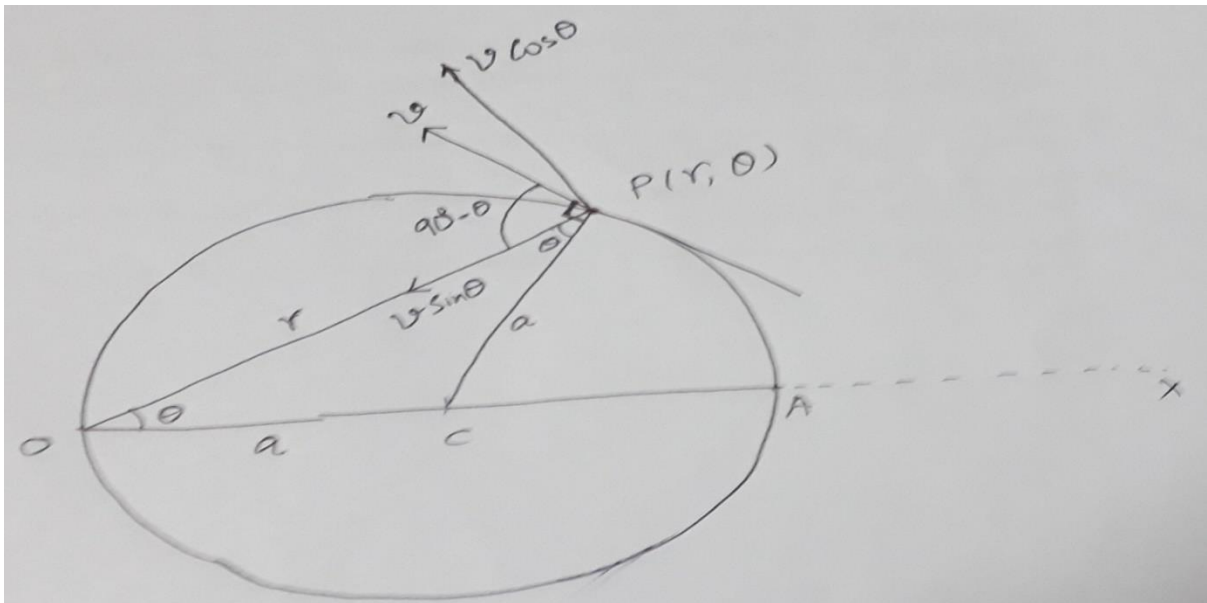
$$\text{i.e. } r = A e^{k\theta}$$

which is an equiangular spiral.

## Example 2

A point moves in a circular path of radius "a" so that its angular velocity about a fixed point in the circumference of the circle is constant, equal to  $w$ . Show that the resultant acceleration of the point at every point of the path is of constant magnitude  $4aw^2$ .

**Solution:** Let 'O' be the fixed point (pole), OC – initial line. Polar equation of the circle is  $r = 2a \cos \theta$ . Let P ( $r, \theta$ ) be the position at time 't' then angular velocity about O is  $\dot{\theta} = w$  (constant)



$$\text{Radial velocity} = \dot{r} = -(2a \sin \theta) \dot{\theta} = -2aw \sin \theta$$

$$\Rightarrow \ddot{r} = -(2aw \cos \theta) \dot{\theta} = -2aw^2 \cos \theta$$

$$= -w^2 (2a \cos \theta)$$

$$= -w^2 \cdot r$$

$$\text{Radial acceleration at P} = \ddot{r} - r(\dot{\theta})^2$$

$$= -w^2 \cdot r - w^2 \cdot r$$

$$= -2w^2 r = -2w^2 (2a \cos \theta)$$

$$= -4aw^2 \cos \theta$$

$$\text{Transverse acceleration at P} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = \frac{1}{r} w \cdot 2r \dot{r}$$

$$= 2 w (-2aw \sin\theta) = -4a \sin\theta$$

$$\text{So, Resultant acceleration} = \sqrt{(-4aw \cos\theta)^2 + (-4aw \sin\theta)^2}$$
$$= 4aw^2$$