SRI JAI NARAIN MISHRA PG COLLEGE LUCKNOW

CO-ORDINATE GEOMETRY

POLAR EQUATION OF A CONIC

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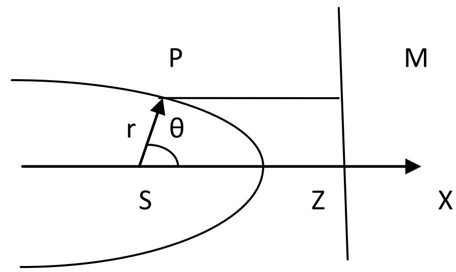
BSc I Semester -2 Paper- 2 Unit-1

Polar Equation of a conic

Definition: The conic is the locus of a point, which moves in such a way that the ratio of its distance from a fixed point to the perpendicular distance from a fixed line remains constant.

The fixed point is called the **focus** and the fixed line is called the **directrix** of the conic. The constant ratio is called the **eccentricity** of the conic.



In adjacent figure,

Fixed point is S (the focus) taken as origin

P is any point on the conic with co-ordinate(r,θ).

PM is perpendicular from P to the fixed line MZ(directrix).

So,

$$\frac{SP}{PM} = e \quad (\text{eccentricity})$$

Polar equation of a conic having length of latus - rectum as 2l and its focus being taken as pole.

Let $P(r,\theta)$ be any point on the conic.

LQ = latus rectum= 2SL=2l

Where SP = r, < PSN = θ, LQ = 2l, S being pole.

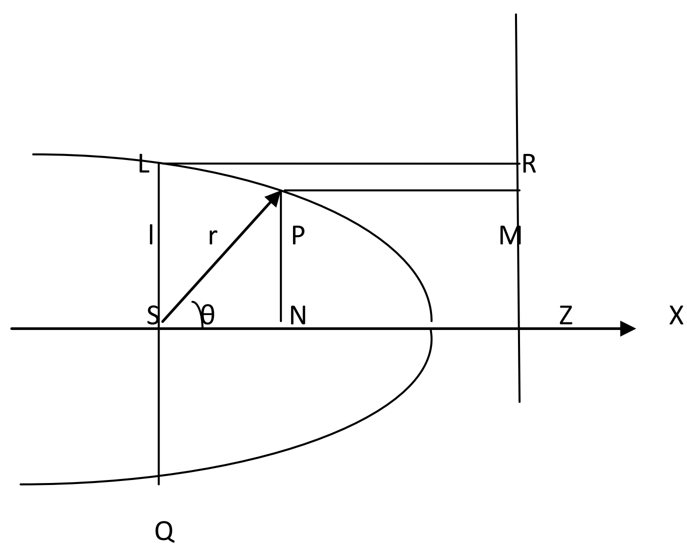
By def. of the conic

$\frac{SP}{PM} = e$ or SP = e . PM or r = e.PM

PM = $\frac{r}{e}$ ...... (1)

By def of conic we also have

$\frac{SL}{LR} = e$ or SL = e.LR or l = e.LR



Or LR = $\frac{l}{e}$ ....(2)

Also LR = SZ = SN + NZ and NZ = PM .In right angled triangle SNP we have SN = r cos θ

LR = r cos θ + $\frac{r}{e}$ or $\frac{l}{e}$ = r cos θ + $\frac{r}{e}$

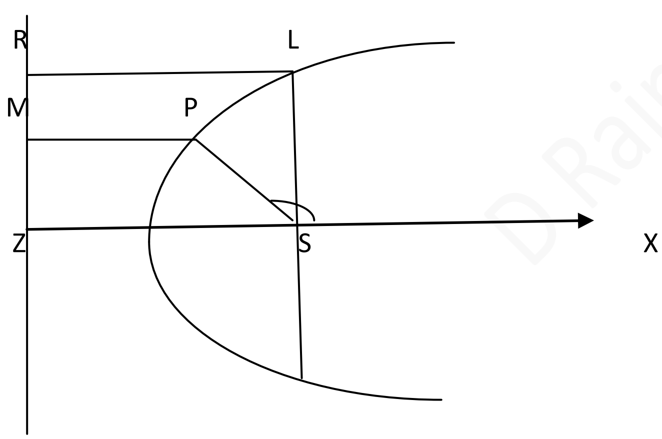
Multiplying the whole equation by $\frac{e}{r}$ we obtain

$$\boxed{\frac{l}{r} = 1 + e\cos\theta} \qquad \text{........... (3)}$$

Which is the the required equation of conic.

Remember:

(i) For the following figure

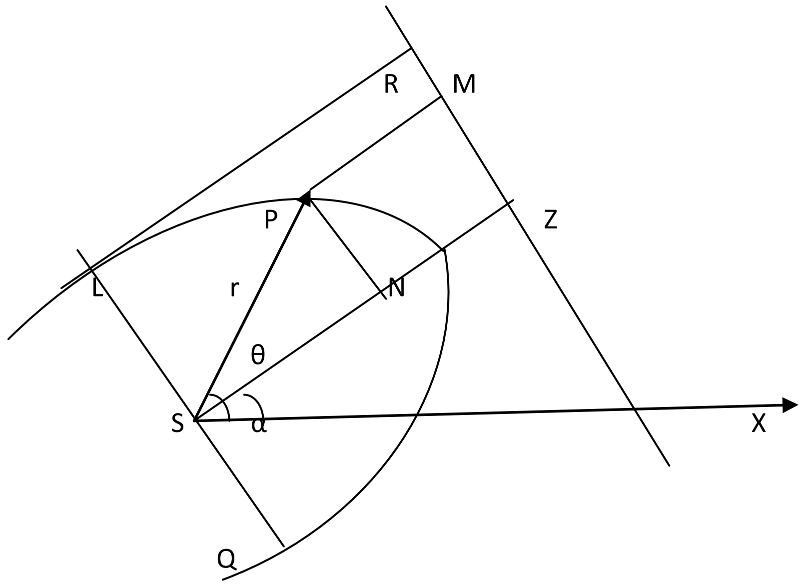


The equation of conic is

$$\frac{l}{r} = 1 - e\cos\theta \qquad \text{..... (4)}$$

We can obtain it by replacing θ by (π + θ)in eqn(3) that means if the initial line is rotated by angle π we get equation (4).

Polar equation of a conic if the axis of the conic is inclined at an angle α with the initial line other conditions remains same.



We can see that the axis of the conic SZ is inclined an angle α with the initial line SX. Let P (r,θ) be a point on the conic.Let,

LQ: the latus rectum of the conic of length 2l. So, SL = semi latus rectum = l , SP = r

and < NSX = α   < PSX = θ   < PSN = θ – α

By def of conic

$$\frac{\mathrm{SP}}{\mathrm{PM}} = \mathrm{e} \qquad \text{and} \qquad \frac{\mathrm{SL}}{\mathrm{LR}} = \mathrm{e}$$

PM = SP/e = r/e   and   SL = e. LR   or   LR = SL / e = l/e ....(5)

From fig we can see that,

LR = SZ = SN + NZ and SN = r cos (θ – α), NZ = PM = r/e ....(6)

Therefore, from (5) and (6) we have

$$\frac{\mathrm{l}}{\mathrm{e}} = \mathrm{r}\cos(\theta - \alpha) + \frac{\mathrm{r}}{\mathrm{e}}$$

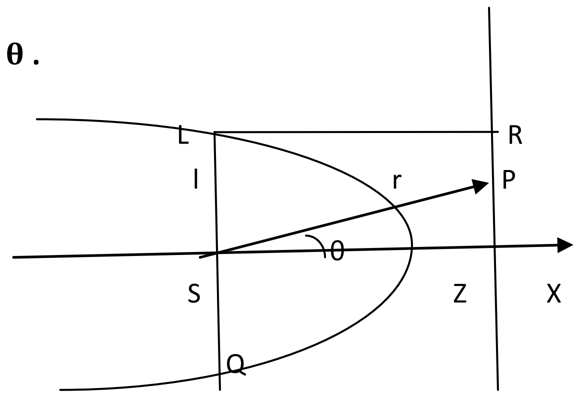
Multiplying it by e/r we have,

$$\boxed{\frac{\mathrm{l}}{\mathrm{r}} = 1 + \mathrm{e}\cos(\theta - \alpha)}$$

Which is the required equation.

Equation of the directrix of a conic $\frac{l}{r} = 1 + e\cos\theta$ **.**

Let RZ be the directrix of the conic near the pole S. We have to find the equation of it.



Let P(r,θ) be a point on the directrix.

SP = r, < PSZ = θ SL = l

From the adjacent fig, in right angled triangle PSZ we have,

SZ = r cos θ, ...(7)

SZ = LR ....(8)

and by def of conic $\frac{SL}{LR} = e$ or LR = SL/e = l/e ....(9)

By eqn (7)(8)(9) we obtain,

$\frac{l}{e} = r\cos\theta$ or

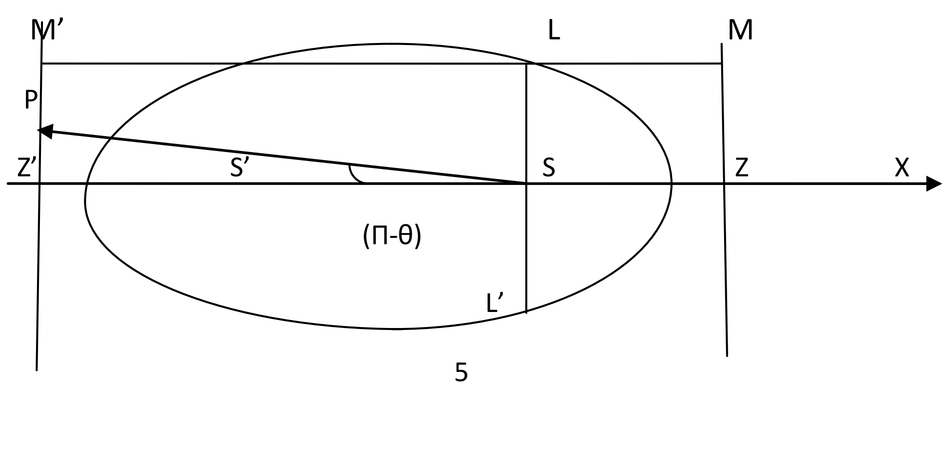
$$\boxed{\frac{l}{r} = e\cos\theta}$$

Which is the required equation of the directrix.

What will be the equations of the directrices if there are two directrices?

We know that there are two directrices in an ellipse. Let S and S' be two foci of the ellipse.Let S be the pole . MZ is the directrix corresponding to the focus S (pole) and M'Z' be the other directrix corresponding to the focus S' .

The directrix MZ has the same equation as obtained above. Now we have to find the equation of the directrix M'Z' with respect to focus (pole) S (Note).



Let

SP = r and < PSZ = θ

LL' : the latus rectum and equal to 2l so SL = l

LM' : the perpendicular from L to the directrix M'Z'.

LM: the perpendicular from L to the directrix MZ.

By def of conic $\frac{SL}{LM'} = e$ and LM' = SL/e = l/e

Also $\frac{SL}{LM} = e$ and LM = SL/e = l/e

Also LM'= SZ'= ZZ' – SZ =ZZ' – LM ....(10) as SZ = LM

We know that in an ellipse,

The sum of distances between the directrices = 2a/e

Length of semi latus rectum $l = \frac{b^2}{a}$, $b^2 = a^2 (1- e^2)$

$l = \frac{a^2}{a} (1 - e^2) = a (1 - e^2)$ or $a = \frac{l}{(1-e^2)}$

and ZZ' = 2a/e = $\frac{2l}{e(1-e^2)}$

Substituting LM and ZZ' in eqn (10), we obtain

SZ' = $\frac{l \; (1+e^2)}{e \; (1-e^2)}$ ....(11)

From figure, SZ' = r cos (π-θ) = - r cos θ ....(12)

From eqn (11) and (12) we have,

$- r \cos\theta = \frac{l \; (1+e^2)}{e \; (1-e^2)}$ on simplifying we get,

$$\frac{l}{r} = -\left(\frac{1-e^2}{1+e^2}\right) e \cos\theta$$

Which is the required equation of the directrix M'Z'.

Equation of the chord joining two points (r_1,θ_1) and (r_2,θ_2) on the conic

$$\frac{l}{r} = 1 + e\cos\theta.$$

The given conic is $\frac{l}{r} = 1 + e\cos\theta$. Let A$(r_1,\theta_1)$ and B(r_2,θ_2) be two points on the conic.

We have to find the equation of the chord AB, joining these points.

Since (r_1,θ_1) and (r_2,θ_2) lie on the conic then

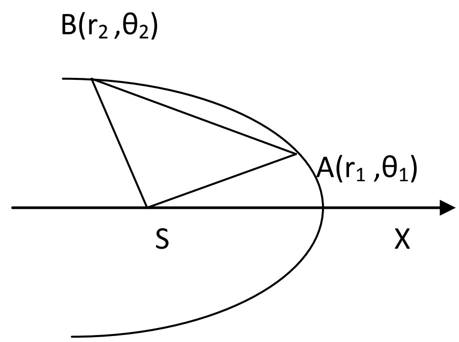
$$\frac{l}{r_1} = 1 + e\cos\theta_1 \quad \text{and} \quad \frac{l}{r_2} = 1 + e\cos\theta_2 \quad \text{..........(13)}$$

Let the polar equation of the line be

$P\cos\theta + Q\sin\theta = l/r$ ....(14)

It passes through the point (r_1,θ_1). Using (13) we have

$P\cos\theta_1 + Q\sin\theta_1 = l/r_1 = 1 + e\cos\theta_1$



$(P - e)\cos\theta_1 + Q\sin\theta_1 - 1 = 0$ ....(15)

Similarly for (r_2,θ_2) we have

$(P - e)\cos\theta_2 + Q\sin\theta_2 - 1 = 0$ ....(16)

Solving (15) and (16) we get,

$$P = \cos\left(\frac{\theta_1+\theta_2}{2}\right)\sec\left(\frac{\theta_2-\theta_1}{2}\right) + e$$

$$Q = \sin\left(\frac{\theta_1+\theta_2}{2}\right)\sec\left(\frac{\theta_2-\theta_1}{2}\right)$$

Substituting these values in eqn(14) and simplifying using trigonometrical formulae we obtain the equation of the chord as

$$\frac{l}{r} = e\cos\theta + \sec\left(\frac{\theta_2-\theta_1}{2}\right)\cos\left\{\theta - \left(\frac{\theta_1+\theta_2}{2}\right)\right\} \quad \text{..........(17)}$$

Equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$ at a given point A(r_1,θ_1).

We know that the tangent at a given point is the limiting position of the chord joining two points. So, continuing from the previous discussion the tangent at the point A is the limiting position of the chord joining the points A(r_1,θ_1) and B(r_2,θ_2) as B → A i .e. $\theta_2 \to \theta_1$ .Taking this limit in the equation of the chord we obtain

$$\boxed{\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_1)} \qquad ....(18)$$

Which is the required equation of the tangent at point A with vectorial angle 'θ_1'.

Remember:(i) The slope of the tangent = - $\left(\frac{e+\cos\theta_1}{\sin\theta_1}\right)$

Which can be obtained by transforming the equation of tangent to cartesian form i.e. by substituting x = r cos θ and y = r sin θ after expanding $\cos(\theta - \theta_1)$ by the trignometrical formula.

(ii) The equation of tangent of the conic $\frac{l}{r} = 1 + e\cos\theta$ at the point with vectorial angle 'θ_1' can be obtained by substituting (π+θ) for θ and (π+θ_1) for θ_1. After simplifying, which comes out to be

$$\boxed{\frac{l}{r} = -e\cos\theta + \cos(\theta - \theta_1)}$$

(iii) The equation of tangent of the conic $\frac{l}{r} = 1 + e\cos(\theta - \alpha)$ at the point 'θ_1' is

$$\boxed{\frac{l}{r} = e\cos(\theta - \alpha) + \cos(\theta - \theta_1)}$$

Equation of the asymptotes of the conic $\frac{l}{r} = 1 + e\cos\theta$.

Let (r_1, θ_1) be a point on the given conic. Then we have

$$\frac{l}{r_1} = 1 + e\cos\theta_1 \qquad ....(18)$$

The equation of tangent to the conic at this point is

$\frac{l}{r_1} = e\cos\theta + \cos\ (\theta - \ \theta_1)$ ....(19)

We know that an asymptote of a conic is the limiting position of its tangent as the point of contact tends to infinity so taking $r_1 \to \infty$ in (18).We get

$0 = 1 + e\cos\theta_1$

Or $\cos\theta_1 = -1/e$

And $\sin\theta_1 = \pm\sqrt{(1 - (1/e^2))}$

From (19) we have

$$\frac{l}{r} = e\cos\theta + \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1$$

Substituting $\cos\theta_1$ and $\sin\theta_1$ in it we get

$$\frac{le}{r} = (e^2 - 1)\cos\theta \pm \sqrt{(e^2-1)}\sin\theta$$

We can see that these asymptotes are real only if $e > 1$ that is given conic is a hyperbola.

Equation of the normal to the conic $\frac{l}{r} = 1 + e\cos\theta$ at the point (r_1 , θ_1').

We know that the equation of the tangent at the point with vectorial angle'θ_1' is

$\frac{l}{r} = e\cos\theta + \cos\ (\theta - \ \theta_1)$

The equation of a straight line perpendicular to it is

$\frac{l'}{r} = e\cos\left(\frac{\pi}{2} + \ \theta\right) + \cos\ \left(\theta + \frac{\pi}{2} - \ \theta_1\right)$

$\frac{l'}{r} = -\ e\sin\theta - \sin\ (\theta - \ \theta_1)$ ....(20)

It passes through the point (r_1 , θ_1') then

$\frac{l'}{r_1} = -\ e\sin\theta_1 - \sin\ (\theta_1 - \ \theta_1)$

$\frac{l'}{r_1} = -\ e\sin\theta_1$ ....(21)

The point (r_1 , θ_1') lies on conic thus we have

$\frac{l}{r_1} = 1 + e\cos\theta_1$ ....(22)

Dividing eqn (20) by (21),we get

$$\frac{l'}{l} = -\left(\frac{e\sin\theta_1}{1+e\cos\theta_1}\right)$$

$$l' = -\left(\frac{e\,l\sin\theta_1}{1+e\cos\theta_1}\right)$$

Substituting l' in eqn (20),

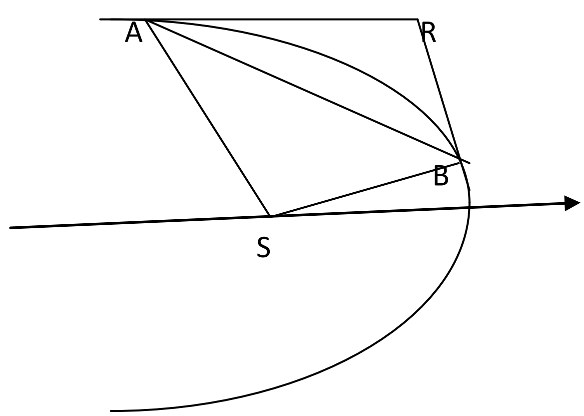
$$-\left(\frac{e\,l\sin\theta_1}{r\,(1+e\cos\theta_1)}\right) = -e\sin\theta - \sin\,(\theta - \theta_1)$$

$$\boxed{\left(\frac{e\,l\sin\theta_1}{r\,(1+e\cos\theta_1)}\right) = e\sin\theta + \sin\,(\theta - \theta_1)} \quad ...(23)$$

Which is the required equation of normal to a conic.

Chord of contact of the point R(r$'$,α) with respect to the conic $\frac{l}{r} = 1 + e\cos\theta$.

Let the tangents drawn from R meet the conic at A and B with vectorial angles 'θ_1' and 'θ_2' respectively.The chord AB is the chord of contact of the point R with respect to the given conic.



The equation of tangent of the conic at A i.e. 'θ_1' is

$\frac{l}{r} = e\cos\theta + \cos\,(\theta - \theta_1)$ ....(24)

The equation of tangent of the conic at B i.e. 'θ_2' is

$\frac{l}{r} = e\cos\theta + \cos\,(\theta - \theta_2)$ ....(25)

Since R(r',α) lies on (24) and (25) therefore we have

$$\frac{l}{r'} = e\cos\alpha + \cos\ (\alpha - \theta_1) \qquad ....(26)$$

$$\frac{l}{r'} = e\cos\alpha + \cos\ (\alpha - \theta_2) \qquad ....(27)$$

Equating (26) and (27) ,

$\alpha - \theta_1 = \pm\ (\alpha - \theta_2)$ if we take +ve sign then we get $\theta_1 = \theta_2$ which is not true.

So we can take $\alpha - \theta_1 = -(\alpha - \theta_2)$

$\therefore\ \alpha = \frac{(\theta_1 + \theta_2)}{2}$,

Substituting this value of α in (27) we get,

$$\frac{l}{r'} = e\cos\alpha + \cos\left(\frac{(\theta_1+\theta_2)}{2} - \theta_2\right)$$

$$\frac{l}{r'} - e\cos\alpha = \cos\ \frac{(\theta_1-\theta_2)}{2} \quad \text{or} \quad \sec\left(\frac{\theta_2-\theta_1}{2}\right) = \frac{1}{\left(\frac{l}{r'} - e\cos\alpha\right)} \qquad ....(28)$$

Using eqn (17) the chord joining A and B is

$$\frac{l}{r} = e\cos\theta + \sec\left(\frac{\theta_2-\theta_1}{2}\right)\cos\left\{\theta - \left(\frac{\theta_1+\theta_2}{2}\right)\right\}$$

Or $\frac{l}{r} - e\cos\theta = \sec\left(\frac{\theta_2-\theta_1}{2}\right)\cos\ (\theta - \alpha)$

Using (28) we obtain,

$$\boxed{\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\alpha\right) = \cos\ (\theta - \alpha)} \qquad ....(29)$$

Which is the required equation of the chord of contact of the tangents drawn from the point R to the conic.

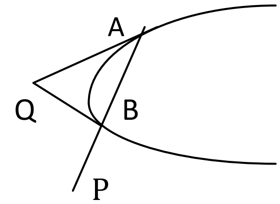
Polar of a point Q(R, γ) with respect to the conic $\frac{l}{r} = 1 + e\cos\theta$.

Let a line through the point P(R,γ) meets the conic at points $A(r_1,\theta_1)$ and $B(r_2,\theta_2)$.Let the tangents to the conic at A and B intersect in the point Q (r′,α). Here we have to find the

locus of Q.We know that equation of chord of contact is

$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\alpha\right) = \cos\ (\theta - \alpha)$$

It passes through the point P(R,γ) we have



$$\left(\frac{l}{R} - e\cos\gamma\right)\left(\frac{l}{r'} - e\cos\alpha\right) = \cos(\gamma - \alpha)$$

∴ The locus of Q(r',α) is

$$\left(\frac{l}{R} - e\cos\gamma\right)\left(\frac{l}{r} - e\cos\theta\right) = \cos(\gamma - \theta)$$

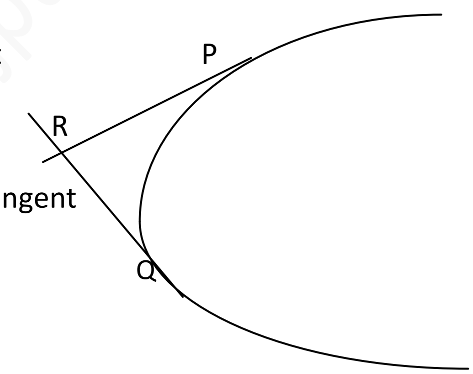
Or

$$\boxed{\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{R} - e\cos\gamma\right) = \cos(\theta - \gamma)} \quad ....(30)$$

Which is the required equation of polar of a point with respect to the conic.

Polar equation of pair of tangents drawn from point R(r',α) to the conic $\frac{l}{r} = 1 + e\cos\theta$.

Let two tangents from P and Q be drawn to the conic Let the extremities of these tangents intersect at the point R(r',α).Let the vectorial angles of point P and Q be 'θ_1' and 'θ_2' respectively.The equation of the tangent at any of these points is



$$\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_1)$$

$$\cos(\theta - \theta_1) = \frac{l}{r} - e\cos\theta \quad ....(31)$$

Since R lies on it

∴ $$\frac{l}{r'} = e\cos\alpha + \cos(\alpha - \theta_1)$$

$$\cos(\alpha - \theta_1) = \frac{l}{r'} - e\cos\alpha \quad ....(32)$$

Now $\cos(\theta - \alpha) = \cos\{(\theta - \theta_1) - (\alpha - \theta_1)\}$

$$= \cos(\theta - \theta_1)\cos(\alpha - \theta_1) + \sin(\theta - \theta_1)\sin(\alpha - \theta_1)$$

$\cos(\theta - \theta_1) - \cos(\theta - \theta_1)\cos(\alpha - \theta_1) = -\sin(\theta - \theta_1)\sin(\alpha - \theta_1)$

Squaring both sides

$(\cos(\theta - \theta_1) - \cos(\theta - \theta_1)\cos(\alpha - \theta_1))^2 = (\sin(\theta - \theta_1)\sin(\alpha - \theta_1))^2$

$= (1 - \cos^2(\theta - \theta_1))(1 - \cos^2(\alpha - \theta_1))$

$$\left\{\cos(\theta - \theta_1) - \left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\alpha\right)\right\}^2 = \left\{1 - \left(\frac{l}{r'} - e\cos\alpha\right)^2\right\}\left\{1 - \left(\frac{l}{r} - e\cos\theta\right)^2\right\}$$

Which gives equation of pair of tangents.

Polar equation of director circle of conic $\frac{l}{r} = 1 + e\cos\theta$.

Definition: The locus of the point of intersection of a pair of perpendicular tangents to a conic is known as the director circle.

Let a pair of tangents be drawn at points P and Q with vectorial angles 'θ_1' and 'θ_2' repectively.

The equations of tangents at P and Q are

The equation of tangent of the conic at A i.e. 'θ_1' is

$\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_1)$ ......(i)

The equation of tangent of the conic at B i.e. 'θ_2' is

$\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_2)$ ....(ii)

Subtracting (ii) from (i) we get

$\cos(\theta - \theta_1) = \cos(\theta - \theta_2)$ or $(\theta - \theta_1) = \pm(\theta - \theta_2)$ taking $(\theta - \theta_1) = -(\theta - \theta_2)$

We get $\theta = \frac{(\theta_1 + \theta_2)}{2}$

Substituting this value of θ in (ii)

$\frac{l}{r} = e\cos\frac{(\theta_1+\theta_2)}{2} + \cos\left(\frac{(\theta_1+\theta_2)}{2} - \theta_2\right) = e\cos\frac{(\theta_1+\theta_2)}{2} + \cos\left(\frac{(\theta_1-\theta_2)}{2}\right)$

The point of intersection(r',α) of tangents (i) and (ii) is given by,

$\alpha = \frac{(\theta_1+\theta_2)}{2}$ and $\frac{l}{r'} = e\cos\frac{(\theta_1+\theta_2)}{2} + \cos\frac{(\theta_1-\theta_2)}{2}$

$\cos\frac{(\theta_1-\theta_2)}{2} = \frac{l}{r'} - e\cos\frac{(\theta_1+\theta_2)}{2} = \frac{l}{r'} - e\cos\alpha$

$\cos\frac{(\theta_1-\theta_2)}{2} = \frac{l}{r'} - e\cos\alpha$

Changing (i) to Cartesian form

$\frac{l}{r} = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1 + e\cos\theta$

$l = r\cos\theta\cos\theta_1 + r\sin\theta\sin\theta_1 + e.\,r\cos\theta$

put $x = r\cos\theta$, $y = r\sin\theta$

$l = x\cos\theta_1 + y\sin\theta_1 + e.\,x = (\cos\theta_1 + e)\,x + y\sin\theta_1$

∴ Slope of tangent (i) is

$m_1 = -(\cos\theta_1 + e)/\sin\theta_1$

Similarly the slope of the other tangent is

$m_2 = -(\cos\theta_2 + e)/\sin\theta_2$

The tangents (i) and (ii) are perpendicular if $m_1 . m_2 = -1$

$(-(\cos\theta_1 + e)/\sin\theta_1) . (-(\cos\theta_2 + e)/\sin\theta_2) = -1$

$(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + e\,(\cos\theta_1 + \cos\theta_2) + e^2 = 0$

$\cos(\theta_1 - \theta_2) + 2e\cos\left(\frac{\theta1+\theta2}{2}\right)\cos\left(\frac{\theta1-\theta2}{2}\right) + e^2 = 0$

$(2\cos^2(\theta_1 - \theta_2) - 1) + 2e\cos\left(\frac{\theta1+\theta2}{2}\right)\cos\left(\frac{\theta1-\theta2}{2}\right) + e^2 = 0$ ....(iii)

Till now we have obtained $\alpha = \left(\frac{\theta1+\theta2}{2}\right)$ and $\cos\frac{(\theta_1-\theta_2)}{2} = \frac{l}{r'} - e\cos\alpha$

Substituting these in (iii)

$2\left(\frac{l}{r'} - e\cos\alpha\right)^2 - 1 + 2e\cos\alpha\left(\frac{l}{r'} - e\cos\alpha\right) + e^2 = 0$

Solving it we get

$(1-e^2)\, r'^2 + 2ler' \cos\alpha - 2l^2 = 0$

The locus of (r',α) is

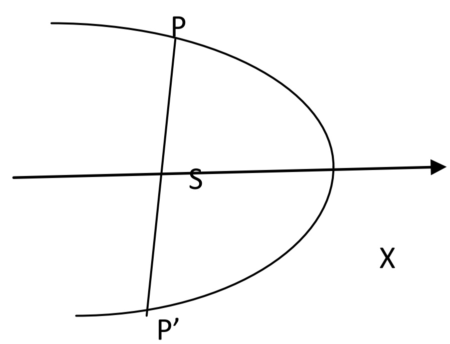
$$(1-e^2)\, r'^2 + 2ler' \cos\alpha - 2l^2 = 0$$

Which is the required equation of the director circle.

Ex. If the PSP' is focal chord of the conic $\frac{3}{r} = 1 + \frac{1}{2}\cos\theta$ whose focus is S, then prove that

$$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{3}$$

Sol. Let the vectorial angle of P and P' be 'θ' and 'π+θ' respectively. Let the conic be



$\frac{l}{r} = 1 + e\cos\theta$ ......(1)

For point P , r = SP

From eqn (1)

$\frac{l}{SP} = 1 + e\cos\theta.$ ....(2)

Similarly for P' we have

$\frac{l}{SP'} = 1 + e\cos(\pi + \theta) = 1 - e\cos\theta$ ......(3)

On comparing the given conic with eqn (1) we get l =3 and e =1/2

Adding (2) and (3)

$$\frac{l}{SP} + \frac{l}{SP'} = 2$$

Put l=3 we get

$\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{3}$ proved.

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